The Complexity of Counting Eulerian Tours in 4-regular Graphs

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Abstract We investigate the complexity of counting Eulerian tours (#ET) and its variations from two perspectives—the complexity of exact counting and the complexity w.r.t. approximation-preserving reductions (AP-reductions, Dyer et al., Algorithmica 38(3):471–500, 2004). We prove that #ET is #P-complete even for planar 4-regular graphs.

A closely related problem is that of counting A-trails (#A-TRAILS) in graphs with rotational embedding schemes (so called maps). Kotzig (Theory of Graphs, Proc. Colloq., Tihany, 1966, pp. 219–230, Academic Press, San Diego, 1968) showed that #A-TRAILS can be computed in polynomial time for 4-regular plane graphs (embedding in the plane is equivalent to giving a rotational embedding scheme). We show that for 4-regular maps the problem is #P-hard. Moreover, we show that from the approximation viewpoint #A-TRAILS in 4-regular maps captures the essence of #ET, that is, we give an AP-reduction from #ET in general graphs to #A-TRAILS in 4-regular maps. The reduction uses a fast mixing result for a card shuffling problem (Wilson, Ann. Appl. Probab. 14(1):274–325, 2004).

Keywords Eulerian tours · A-trails · #P-complete · AP-reduction · Fully polynomial randomized approximation scheme

1 Introduction

An Eulerian tour in a graph is a tour which travels each edge exactly once. The problem of counting Eulerian tours (#ET) of a graph is one of a few recognized counting problems (see, e.g., [15], p. 339). The exact counting is #P-complete in general graphs [4] and in planar graphs [5], and thus there is no polynomial-time algorithm for it unless P = NP. For the approximate counting one wants to have a fully polynomial randomized approximation scheme (FPRAS), that is, an algorithm which on every instance x of the problem and error parameter $\varepsilon > 0$, will output a value within a factor $\exp(\pm\varepsilon)$ of $f(x)$ with probability at least $2/3$ and in time polynomial in the length of the encoding of x and $1/\varepsilon$, where $f(x)$ is the value we want to compute. The existence of an FPRAS for #ET is an open problem [11, 14, 15].

A closely related problem to #ET is the problem of counting A-trails (#A-TRAILS) in graphs with rotational embedding schemes (called maps, see Sect. 2 for a definition). A-trails were studied in the context of decision problems (for example, it is NP-complete to decide whether a given plane graph has an A-trail [1, 3]; on the other hand for 4-regular maps the problem is in P [6]), as well as counting problems (for example, Kotzig [13] showed that #A-TRAILS can be computed in polynomial time for 4-regular plane graphs, reducing the problem to counting of spanning trees).

In this paper, we investigate the complexity of #ET in 4-regular graphs and its variations from two perspectives. First, the complexity of exact counting is considered. We prove that #ET in 4-regular graphs (even in 4-regular planar graphs) is #P-complete. We also prove that #A-TRAILS in 4-regular maps is #P-complete (recall that the problem can be solved in polynomial time for 4-regular *plane* graphs).

The second perspective is the complexity w.r.t. the AP-reductions proposed by Dyer, Goldberg, Greenhill, and Jerrum [7]. We give an AP-reduction from #ET in general graphs to #A-TRAILS in 4-regular maps. Thus we show that if there is an FPRAS for #A-TRAILS in 4-regular maps, then there is also an FPRAS for #ET in general graphs. The existence of AP-reduction from #ET in general graphs to #ET in 4-regular graphs is left open.

In order to understand whether #A-TRAILS in 4-regular maps can AP-reduce to #ET in 4-regular graphs, we investigate the so-called signatures (these count connection patterns of trails in graphs with half-edges, see Sect. 5 for the formal definition) of 4-regular map gadgets and 4-regular graph gadgets. It seems that the signatures represented by 4-regular map gadgets form a proper superset of the set of signatures represented by 4-regular graph gadgets. Moreover, it seems that the signature of a single vertex in 4-regular maps cannot be simulated approximately by 4-regular graph gadgets.

2 Definitions and Terminology

For the definitions of cyclic orderings, A-trails, and mixed graphs, we follow [8]. Let $G = (V, E)$ be a graph. For a vertex $v \in V$ of degree $d > 0$, let $K(v) = \{e_1, \ldots, e_d\}$ be the set of edges adjacent to v in G. The *cyclic ordering* $O^+(v)$ of the edges adjacent to v is a d-tuple $(e_{\sigma(1)}, \ldots, e_{\sigma(d)})$, where σ is a permutation in S_d. We

say $e_{\sigma(i)}$ and $e_{\sigma(i+1)}$ are cyclicly-adjacent in $O^+(v)$, for $1 \le i \le d$, where we set $\sigma(d+1) := \sigma(1)$. The set $O^+(G) = \{O^+(v) | v \in V\}$ is called a *rotational embedding scheme* of G. For a plane graph $G = (V, E)$, if $O^+(v)$ is not specified, we usually set $O^+(v)$ to be the clockwise order of the half-edges adjacent to v for each $v \in V$.

Let $G = (V, E)$ be a graph with a rotational embedding scheme $O^+(G)$. An Eulerian tour $v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell = v_0$ is called an *A-trail* if e_i and e_{i+1} are cyclicly-adjacent in $O^+(v_i)$, for each $1 \le i \le \ell$, where we set $e_{\ell+1} := e_1$.

Let $G = (V, E, E')$ be a mixed graph, that is, E is the set of edges and E' is the set of half-edges (which are incident with only one vertex in V). Let $|E'| = 2d$ where d is a positive integer and assume that the half-edges in E' are labelled by numbers from 1 to $2d$. A *route* $r(a, b)$ is a trail (no repeated edges, repeated vertices allowed) in G that starts with half-edge a and ends with half-edge b. A collection of d routes is called *valid* if every edge and every half-edge is travelled exactly once.

We say that a valid set of routes is of the *type* $\{\{a_1, b_1\}, \ldots, \{a_d, b_d\}\}$ if it contains routes connecting a_i to b_i for $i \in [d]$. We use $\mathrm{VR}(\{a_1, b_1\}, \ldots, \{a_d, b_d\})$ to denote the set of valid sets of routes of type $\{\{a_1, b_1\}, \ldots, \{a_d, b_d\}\}$ in G.

We will use the following concepts from Markov chains to construct the gadget in Sect. 4 (see, e.g., [12] for more detail). Given two probability distributions π and π' on finite set Ω, the *total variation distance* between π and π' is defined as

$$\|\pi - \pi'\|_{TV} = \frac{1}{2} \sum_{\omega \in \Omega} |\pi(\omega) - \pi'(\omega)| = \max_{A \subseteq \Omega} |\pi(A) - \pi'(A)|.$$

Given a finite ergodic Markov chain with transition matrix P and stationary distribution π, the *mixing time* from initial state x, denoted as $\tau_x(\varepsilon)$, is defined as $\tau_x(\varepsilon) = \min\{t : \|P^t(x, \cdot) - \pi\|_{TV} \le \varepsilon\}$, and the mixing time of the chain $\tau(\varepsilon)$ is defined as $\tau(\varepsilon) = \max_{x \in \Omega}\{\tau_x(\varepsilon)\}$.

3 The Complexity of Exact Counting

3.1 Basic Gadgets

We describe two basic gadgets and their properties which will be used as a basis for larger gadgets in the subsequent sections.

The first gadget, which is called the (X, Y, Y) node, is shown in Fig. 1(a), and it is represented by the symbol shown in Fig. 1(b). There are k internal vertices in the gadget, and the labels 0, 1, 2 and 3 are four half-edges of the (X, Y, Y) node which are the only connections from the outside.

By elementary counting we obtain the following fact.

Lemma 1 *The (X, Y, Y) node with parameter k has three different types of valid sets of routes and these satisfy*

$$|\mathrm{VR}(\{0, 1\}, \{2, 3\})| = k2^{k-1},$$
$$|\mathrm{VR}(\{0, 2\}, \{1, 3\})| = |\mathrm{VR}(\{0, 3\}, \{1, 2\})| = 2^{k-1}.$$

The gadget has k vertices.

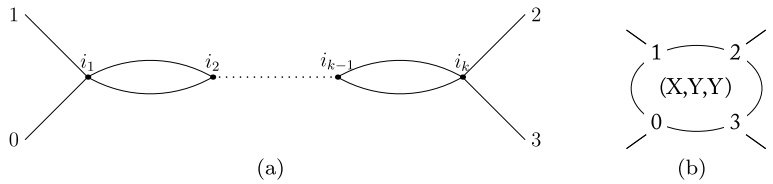


Fig. 1 An (X, Y, Y) node and its symbol. (**a**): an (X, Y, Y) node consisting of k internal vertices; (**b**): symbol representing the (X, Y, Y) node

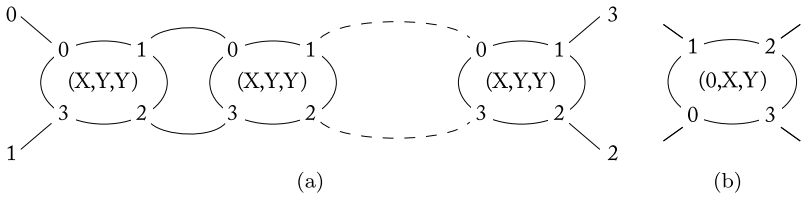


Fig. 2 A $(0, X, Y)$ node and its symbol. (**a**): a $(0, X, Y)$ node consisting of p copies of (X, Y, Y) nodes; (**b**): symbol representing the $(0, X, Y)$ node

The second gadget, which is called the $(0, X, Y)$ node, is shown in Fig. 2(a), and it is represented by the symbol shown in Fig. 2(b). Let p be any odd prime. In the construction of the $(0, X, Y)$ node we use p copies of (X, Y, Y) nodes as basic components, and each (X, Y, Y) node has the same parameter k. As illustrated, half-edges are connected between two consecutive (X, Y, Y) nodes. The four labels 0, 1, 2 and 3 at four corners in Fig. 2(a) are the four half-edges of the $(0, X, Y)$ node, and they are the only connections from the outside.

By elementary counting, binomial expansion, Fermat's little theorem, and the fact that 2 has a multiplicative inverse (modulo p), we obtain the following:

Lemma 2 *Let p be an odd prime and let k be an integer. The $(0, X, Y)$ node with parameters p and k has three different types of valid sets of routes and these satisfy*

$$|\mathrm{VR}(\{0,1\},\{2,3\})| = pA(A+B)^{p-1} \equiv 0 \pmod{p}, \tag{1}$$

$$|\mathrm{VR}(\{0,2\},\{1,3\})| = \frac{(A+B)^p - (B-A)^p}{2} \equiv A \pmod{p}, \tag{2}$$

$$|\mathrm{VR}(\{0,3\},\{1,2\})| = \frac{(A+B)^p + (B-A)^p}{2} \equiv B \pmod{p}, \tag{3}$$

where $A = 2^{k-1}$ and $B = k2^{k-1}$. The gadget has kp vertices.

3.2 #ET in 4-regular Graphs is #P-complete

Next, we will give a reduction from #ET in general Eulerian graphs to #ET in 4-regular graphs.

Theorem 1 #ET *in general Eulerian graphs is polynomial time Turing reducible to* #ET *in 4-regular graphs.*

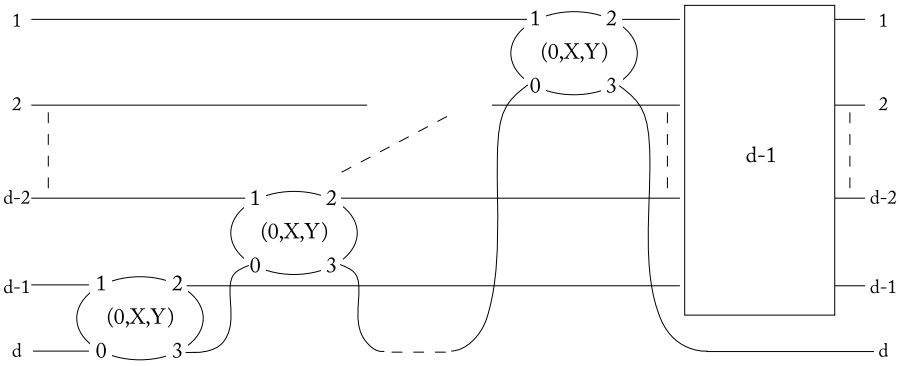


Fig. 3 Gadget Q with d input half-edges and d output half-edges

The proof of Theorem 1 is postponed to the end of this section.

We use the gadget, which we will call Q, illustrated in Fig. 3 to prove the theorem. The gadget is constructed in a recursive way. The d labels $1, \ldots, d$ on the left are called input half-edges of the gadget, and the d labels on the right are called output half-edges. Given a prime p and a positive integer d, the gadget consists of $d-1$ copies of $(0, X, Y)$ nodes with different parameters and one recursive part represented by a rectangle with $d-1$ input half-edges and $d-1$ output ones. For $1 \leq i \leq d-1$, the i-th $(0, X, Y)$ node from left has parameters p and i. Half-edge 0 of the i-th $(0, X, Y)$ node is connected to half-edge 3 of the $(i-1)$-st $(0, X, Y)$ node except that for the 1st $(0, X, Y)$ node half-edge 0 is the d-th input half-edge of the gadget. Half-edge 1 of the i-th $(0, X, Y)$ node is the $(d-i)$-th input half-edge of the gadget. Half-edge 2 of the i-th $(0, X, Y)$ node is connected to the $(d-i)$-th input half-edge of the rectangle. Half-edge 3 of the $(d-1)$-st $(0, X, Y)$ node is the d-th output half-edge of the gadget. For $1 \leq j \leq d-1$, the j-th output half-edge of the rectangle is the j-th output half-edge of the gadget. From the constructions of (X, Y, Y) nodes and $(0, X, Y)$ nodes, the total size of the $d-1$ copies of $(0, X, Y)$ nodes is $O(pd^2)$. Thus, the size of the gadget is $O(pd^3)$.

Lemma 3 *Consider the gadget Q with parameters d and p. Let σ be a permutation in S_d. Then*

$$|\mathrm{VR}(\sigma)| := |\mathrm{VR}(\{IN_1, OUT_{\sigma(1)}\}, \ldots, \{IN_d, OUT_{\sigma(d)}\})| \equiv R_d \pmod{p}, \quad (4)$$

where $R_d \equiv \prod_{i=1}^{d-1} (2^{i(i-1)/2} i!)$.

Moreover, any type τ which connects two IN (or two OUT) half-edges satisfies

$$|\mathrm{VR}(\tau)| \equiv 0 \pmod{p}. \quad (5)$$

Proof The proof is by induction on d, the base case $d = 1$ is trivial. Suppose the statement is true for gadget Q with $(d-1)$ input half-edges, that is, $|\mathrm{VR}(\varrho)| \equiv R_{d-1} \pmod{p}$ for every $\varrho \in S_{d-1}$.

Now, consider gadget Q with d input half-edges. For $1 \le j \le d-1$, we cut the gadget by a vertical line just after the j-th $(0, X, Y)$ node and only consider the part of the gadget to the left of the line, we will call this partial gadget Q_j.

Claim 1 *Let A_s be the set of permutations in S_d which map s to d. In the partial gadget Q_j we have that for $s \in \{d-j, \ldots, d\}$ have*

$$\sum_{\sigma \in A_s} |\mathrm{VR}_{Q_j}(\sigma)| \equiv j! 2^{j(j-1)/2} \pmod{p},$$

where the subscript Q_j is used to indicate that we count routes in gadget Q_j.

Proof of Claim We prove the claim by induction on j, the base case $j = 1$ is trivial.

Now assume that the claim is true for $j-1$, that is, for all $s \in \{d-j+1, \ldots, d\}$ in gadget Q_{j-1} we have

$$\sum_{\sigma \in A_s} |\mathrm{VR}_{Q_{j-1}}(\sigma)| \equiv (j-1)! 2^{(j-1)(j-2)/2} \pmod{p}.$$

The j-th $(0, X, Y)$ node takes $(d-j)$-th input half-edge of the gadget and the half-edge 3 of the $(j-1)$-st $(0, X, Y)$ node, and has parameters p and j.

The type of the j-th $(0, X, Y)$ node is $\{\{0, 2\}, \{1, 3\}\}$ if and only if the resulting permutation in Q_j is in A_{d-j}. Thus we have

$$\sum_{\sigma \in A_{d-j}} |\mathrm{VR}_{Q_j}(\sigma)| \equiv 2^{j-1} \prod_{k=1}^{j-1} (2^{k-1}(k+1)) \equiv j! 2^{j(j-1)/2} \pmod{p},$$

where the first term is the number of choices (modulo p) in the j-th $(0, X, Y)$ node to make it $\{\{0, 2\}, \{1, 3\}\}$ and the k-th term in the product is the number of choices (modulo p) in the k-th $(0, X, Y)$ node to make it either $\{\{0, 2\}, \{1, 3\}\}$ or $\{\{0, 3\}, \{1, 2\}\}$.

If the type inside the j-th $(0, X, Y)$ node is $\{\{0, 3\}, \{1, 2\}\}$ then the resulting permutation is in A_s for $s \in \{d-j+1, \ldots, d\}$. Thus

$$\sum_{\sigma \in A_s} |\mathrm{VR}_{Q_j}(\sigma)| \equiv j 2^{j-1} \sum_{\sigma \in A_s} |\mathrm{VR}_{Q_{j-1}}(\sigma)| \equiv j 2^{j-1} (j-1)! 2^{(j-1)(j-2)/2}$$

$$\equiv j! 2^{j(j-1)/2} \pmod{p},$$

where $j 2^{j-1}$ is the number of choices (modulo p) in the j-th $(0, X, Y)$ node to make it $\{\{0, 3\}, \{1, 2\}\}$. □

Now we continue with the proof of Lemma 3.

Let σ be a permutation in S_d. Let $l = \sigma^{-1}(d)$. In order for σ to be realized by gadget Q we have to have l mapped to d by Q_{d-1} and the permutation realized by the recursive gadget of size $d-1$ must "cancel" the permutation of Q_{d-1}. By the claim there are $(d-1)! 2^{(d-1)(d-2)/2}$ (modulo p) choices in Q_{d-1} which map l to d and by the inductive hypothesis there are R_{d-1} (modulo p) choices in the recursive

gadget of size $d-1$ that give the unique permutation that "cancels" the permutation of Q_{d-1}. Thus

$$|\mathrm{VR}(\sigma)| \equiv R_d \equiv (d-1)!2^{(d-1)(d-2)/2} R_{d-1} (\mathrm{mod}\ p),$$

finishing the proof of (4).

To see (5) note that the number of valid sets of routes which contain route starting and ending at both input half-edges or both output half-edges is 0 (modulo p). This is because the number of valid set of routes of type $\{\{0,1\},\{2,3\}\}$ inside the $(0,X,Y)$ node is 0 (modulo p). □

Proof of Theorem 1 The reduction is now a standard application of the Chinese remainder theorem. Given an Eulerian graph $G=(V,E)$, we can, w.l.o.g., assume that the degree of vertices of G is at least 4 (vertices of degree 2 can be removed by contracting edges). The number of Eulerian tours of a graph on n vertices is bounded by n^{n^2} (the number of pairings in a vertex of degree d is $d!/(2^{d/2}(d/2)!) \leq n^n$).

We choose n^2 primes $p_1,\ldots,p_{n^2} > n$ such that $\prod_{i=1}^{n^2} p_i > n^{n^2}$ and each p_i is bounded by $O(n^3)$ (see, e.g., [2], p. 296). For each p_i, we construct graph G_i by replacing each vertex v of degree $d>4$ with Q gadget with d input and d output half-edges where the $(2j-1)$-st and $2j$-th output half-edge are connected (for $j=1,\ldots,d/2$), and the input half-edges are used to replace half-edges emanating from v (that is, they are connected to the input half-edges of other gadgets according to the edge incidence at v). Note that G_i is a 4-regular graph. Since $p_i = O(n^3)$, the construction of G_i can be done in time polynomial in n. Having G_i, we make a query to the oracle and obtain the number T_i of Eulerian tours in G_i. Let T be the number of Eulerian tours in G. Then

$$T_i \equiv T \prod_{d=6}^{n} \left(\left(\frac{d}{2} \right)! 2^{d/2} R_d \right)^{n_d} (\mathrm{mod}\ p_i), \tag{6}$$

where n_d is the number of vertices of degree d in G.

Since T_i is of length polynomial in n, we can compute T_i (mod p_i) for each i and thus T (mod p_i) (since on the right hand side of (6) T is multiplied by a term that has an inverse modulo p_i). By the Chinese remainder theorem, we can compute T in time polynomial in n (see, e.g., [2], p. 106). □

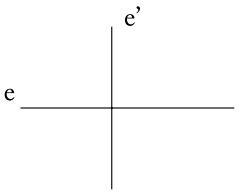
3.3 #ET in 4-regular Planar Graphs is #P-complete

First, it's easy to see that #ET in 4-regular planar graphs is in #P. We will give a reduction from #ET in 4-regular graphs to #ET in 4-regular planar graphs.

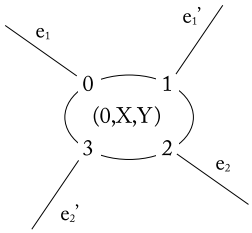
Theorem 2 #ET *in* 4*-regular graphs is polynomial time Turing reducible to* #ET *in* 4*-regular planar graphs.*

Proof Given a 4-regular graph $G=(V,E)$, we first draw G in the plane. We allow the edges to cross other edges, but (i) edges do not cross vertices, (ii) each crossing involves 2 edges. The embedding can be found in polynomial time.

Fig. 4 To replace a crossover point by a $(0, X, Y)$ node with parameters p and $k = p$



(a) Two crossing edges



(b) After the replacement

Let p be an odd prime, we will construct a graph G_p from the embedded graph as follows. Let e, e' be two edges in G which cross in the plane as shown in Fig. 4(a), we split e (and e') into two half-edges e_1, e_2 (e'_1, e'_2, respectively). As illustrated in Fig. 4(b), a $(0, X, Y)$ node with parameters p and $k = p$ is added, and e_1, e'_1, e_2, e'_2 are connected to the half-edges 0, 1, 2, 3 of the $(0, X, Y)$ node, respectively.

Let G_p be the graph after replacing all crossings by $(0, X, Y)$ nodes. We have that G_p is planar since (X, Y, Y) nodes and $(0, X, Y)$ nodes are all planar. The construction can be done in time polynomial in p and the size of G (since the number of crossover points is at most $O(|E|^2)$ and the size of each $(0, X, Y)$ node is $O(p^2)$).

In the reduction, we choose $n = |V|$ primes $p_1, p_2, \ldots, p_n$ such that $p_i = O(n^2)$ for $i \in [n]$ and $\prod_{i=1}^{n} p_i \geq 3^n$, where 3^n is an upper bound for the number of Eulerian tours in G (the number of pairings in each vertex is 3). For each p_i, we construct a graph G_{p_i} from the embedded graph as described above with $p = p_i$. Let T be the number of Eulerian tours in G and T_i be the number of Eulerian tours in G_{p_i}, we have

$$T \equiv T_i \pmod{p_i}. \tag{7}$$

Equation (7) follows from the fact that the number of Eulerian tours in which the set of routes within any $(0, X, Y)$ node is not of type $\{\{0, 2\}, \{1, 3\}\}$ is zero (modulo p_i) (since in (2) we have $A \equiv 1 \pmod{p_i}$ and in (3) we have $B \equiv 0 \pmod{p_i}$). We can make a query to the oracle to obtain the number T_i. By the Chinese remainder theorem, we can compute T in time polynomial in n. □

3.4 #A-TRAILS in 4-regular Graphs with Rotational Embedding Schemes Is #P-complete

In this section, we consider #A-TRAILS in graphs with rotational embedding schemes (maps). We prove that #A-TRAILS in 4-regular maps is #P-complete by a simple reduction from #ET in 4-regular graphs.

First, it's not hard to verify that #A-TRAILS in 4-regular maps is in #P.

Theorem 3 #ET *in* 4*-regular graphs is polynomial time Turing reducible to* #A-TRAILS *in* 4*-regular maps.*

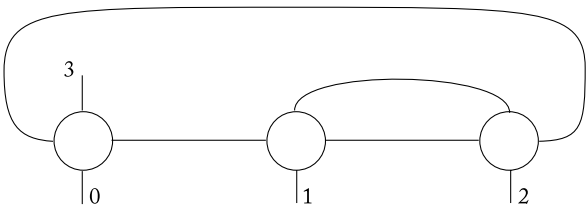


Fig. 5 Gadget simulating vertex of degree 4

Proof Given a 4-regular graph $G = (V, E)$, for each vertex v of G, we use the gadget shown in Fig. 5 to replace v.

The gadget consists of three vertices which are represented by circles in Fig. 5. The labels 0, 1, 2 and 3 are the four half-edges which are used to replace half-edges emanating from v. The cyclic ordering of the 4 (half-)edges incident to each circle is given by the clockwise order, as shown in Fig. 5. There are three types of valid sets of routes inside the gadget, VR($\{0, 1\}, \{2, 3\}$), VR($\{0, 2\}, \{1, 3\}$) and VR($\{0, 3\}, \{1, 2\}$). By enumeration, we have the size of each of the three sets is 2.

Let G' be the 4-regular map obtained by replacing each vertex v by the gadget. Let T be the number of Eulerian tours in G, we have the number of A-trails in G' is $2^{|V|}T$. □

Note that Kotzig [13] gave a one-to-one correspondence between the A-trails in any 4-regular plane graph G (the embedding in the plane gives the rotational embedding scheme) and the spanning trees in a plane graph G', where G is the medial graph of G'. By the Kirchhoff's theorem (cf. [12]), the number of spanning trees of any graph can be computed in polynomial time. Thus #A-TRAILS in 4-regular plane graphs can be computed in polynomial time.

4 The Complexity of Approximate Counting

In this section, we show that #ET in general graphs is AP-reducible to #A-TRAILS in 4-regular maps. AP-reductions were introduced by Dyer, Goldberg, Greenhill and Jerrum [7] for the purpose of comparing the complexity of two counting problems in terms of approximation (given two counting problems f, g, if f is AP-reducible to g and there is an FPRAS for g, then there is also an FPRAS for f).

In the AP-reduction from #ET to #A-TRAILS in 4-regular maps, we use the idea of simulating the pairings in a vertex by a gadget as what we did in the construction of the Q gadget. The difference is that the new gadget works in an approximate way, that is, instead of having the number of valid sets of routes to be the same for each of the types, the numbers can be different but within a small multiplicative factor. The analysis of the gadget uses a fast mixing result for a card shuffling problem.

We use the gadget illustrated in Fig. 6. The circles represent the vertices in the map. Let d be an even number. The gadget has d input half-edges on left and d output half-edges (Fig. 6 demonstrates the case of $d = 6$). There are T layers in the gadget which are numbered from 1 to T from left to right. In an odd layer t, the

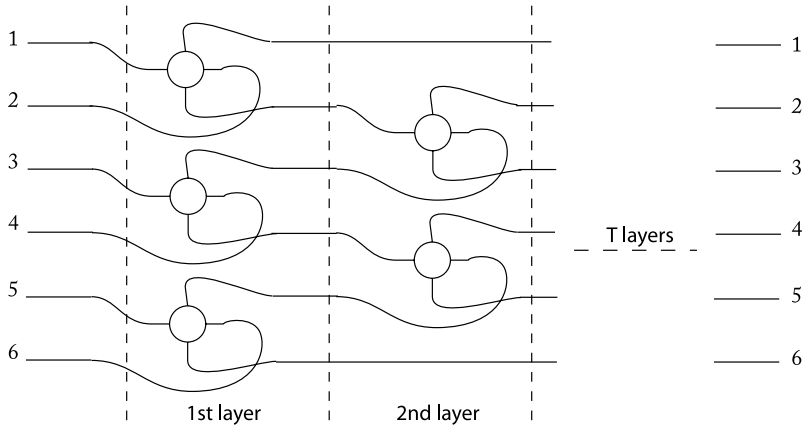


Fig. 6 Construction of the gadget for a vertex of degree 6

$(2i-1)$-st and the $2i$-th output half-edges of layer $t-1$ are connected to a vertex of degree 4, for $i \in [d/2]$. In an even layer t, the $2i$-th and the $(2i+1)$-st output half-edges of layer $t-1$ are connected to a vertex of degree 4, for $i \in [d/2-1]$. In Fig. 6, we illustrate the first two layers each of which is in two consecutive vertical dashed lines. The cyclic ordering of each vertex is given by the clockwise ordering (in the drawing in Fig. 6), and so we have that the two half-edges in each vertex which are connected to half-edges of the previous layer are not cyclicly-adjacent.

Note that a valid route in the gadget always connects an input half-edge to an output half-edge. Thus a valid set of routes always realizes some permutation σ connecting input half-edge i to output half-edge $\sigma(i)$.

In order to prove that $|\mathrm{VR}(\sigma)|$ is almost the same for each permutation $\sigma \in S_d$, we show that for $T = \Theta(d^2 \log d \log(d!/\varepsilon))$ we have

$$|\mathrm{VR}(\sigma)| \Big/ \sum_{\varrho \in S_d} |\mathrm{VR}(\varrho)| \in [(1-\varepsilon)/d!, (1+\varepsilon)/d!]$$

for each permutation $\sigma \in S_d$. The gadget can be interpreted as a process of a Markov chain for shuffling d cards. The simplest such chain proceeds by applying adjacent transpositions. The states of the chain are all the permutations in S_d. In each time step, let $\sigma \in S_d$ be the current state, we choose $i \in \{1, \ldots, d-1\}$ uniformly at random, and then switch $\sigma(i)$ and $\sigma(i+1)$ with probability $1/2$ and stay the same with probability $1/2$. For our gadget, it can be viewed as an even/odd sweeping Markov chain on d cards [16]. The ratio $|\mathrm{VR}(\sigma)|/\sum_{\varrho \in S_d} |\mathrm{VR}(\varrho)|$ is exactly the probability of being σ at time T when the initial state of the even/odd sweeping Markov chain is the identity permutation. By the analysis in [16], we can relate T with the ratio as follows.

Lemma 4 [16] *Let T be the number of layers of the gadget with d input half-edges and d output half-edges as shown in Fig. 6, and let μ, λ be two distributions on S_d such that $\mu(\sigma) = |\mathrm{VR}(\sigma)|/\sum_{\varrho \in S_d} |\mathrm{VR}(\varrho)|$ and $\lambda(\sigma) = 1/d!$ (λ is the uniform*

distribution on S_d). For

$$T = O(d^2 \log d \log(d!/\varepsilon)),$$

then $\|\mu - \lambda\|_{TV} \le \varepsilon/d!$, *and thus* $(1-\varepsilon)/d! \le \mu(\sigma) \le (1+\varepsilon)/d!$.

Theorem 4 *If there is an FPRAS for* #A-TRAILS *in* 4*-regular maps, then we have an FPRAS for* #ET *in general graphs.*

Proof Given an Eulerian graph $G = (V, E)$ and an error parameter $\varepsilon > 0$, we can, w.l.o.g., assume that the degree of vertices of G is at least 4 (vertices of degree 2 can be removed by contracting edges). We construct graph G' by replacing each vertex v of degree $d > 2$ with a gadget with d input half-edges, d output half-edges and $T_d = \Theta(d^2 \log d \log(4d!n/\varepsilon))$ layers where the $(2i-1)$-st and $2i$-th output half-edge are connected (for $1 \le i \le d/2$), and the input half-edges are used to replace half-edges emanating from v (that is, they are connected to the input half-edges of other gadgets according to the edge incidence at v). We have that G' has $O(n^2 T_n) = O(n^4 \log n(n \log n + \log(1/\varepsilon)))$ vertices and can be constructed in time polynomial in n and $1/\varepsilon$.

Let $\mathcal{A}$ be an FPRAS for #A-TRAILS in 4-regular maps by the assumption of the theorem, we run $\mathcal{A}$ on G' with error parameter $\varepsilon/2$. Let $\mathcal{A}(G', \varepsilon/2)$ be the output of $\mathcal{A}$ and N_A be the number of A-trails in G', we have $\mathcal{A}(G', \varepsilon/2) \in [e^{-\varepsilon/2} N_A, e^{\varepsilon/2} N_A]$ with probability at least 2/3. This process can be done in time polynomial in the size of G' and $1/\varepsilon$, which is polynomial in n and $1/\varepsilon$.

Let D_d be the number of vertices in the gadget of d input half-edges and d output half-edges, and let $R_d = 2^{D_d} 2^{d/2} (d/2)!/d!$ and $R = \prod_{d=4}^{n} R_d^{n_d}$ where n_d is the number of vertices of degree d in G. Our algorithm $\mathcal{B}$ will output

$$\mathcal{B}(G, \varepsilon) = \mathcal{A}\big(G', \varepsilon/2\big)/R. \tag{8}$$

We next prove that $\mathcal{B}$ is an FPRAS for #ET in general graphs. For every Eulerian tour in G, the type of the pairing in each vertex in G is fixed. Note that each pairing corresponds to $(d/2)!2^{d/2}$ permutations in a gadget with d input half-edges and d output half-edges. By Lemma 4, we have

$$(1 - \varepsilon/(4n)) 2^{D_d}/d! \le |\mathrm{VR}(\sigma)| \le (1 + \varepsilon/(4n)) 2^{D_d}/d!$$

for each $\sigma \in S_d$ where $\mathrm{VR}(\sigma)$ is counted in a gadget with d input half-edges and d output half-edges. Thus, the number of A-trails in G' which correspond to the same Eulerian tour in G is in $[(1-\varepsilon/(4n))^n R, (1+\varepsilon/(4n))^n R]$. Let N_E be the number of Eulerian tours in G, we have

$$N_A \in [(1 - \varepsilon/(4n))^n R N_E, (1 + \varepsilon/(4n))^n R N_E],$$

and thus for $\varepsilon \le 2n$, $N_A/R \in [e^{-\varepsilon/2} N_E, e^{\varepsilon/4} N_E]$ (the case when $\varepsilon > 2n$ is trivial, $\mathcal{B}$ can just output 3^n). Since $\mathcal{A}(G', \varepsilon/2) \in [e^{-\varepsilon/2} N_A, e^{\varepsilon/2} N_A]$ with probability at least 2/3, then by (8), we have $\mathcal{B}(G, \varepsilon) \in [e^{-\varepsilon} N_E, e^{\varepsilon} N_E]$ with probability at least 2/3. This completes the proof. □

5 The Power of 4-regular Gadgets

In this section, we consider 4-regular gadgets which are 4-regular graphs (or maps) with 4 half-edges (which are labeled from 0 to 3 and are the only connection from outside). There are three types of valid sets of routes inside the gadget, VR($\{0,1\},\{2,3\}$), VR($\{0,2\},\{1,3\}$) and VR($\{0,3\},\{1,2\}$). Since we are interested in the relative size of the above three sets, we define the *signature* of a gadget to be a triple (α,β,γ) such that

$$\alpha = |\mathrm{VR}(\{0,1\},\{2,3\})|/N,$$
$$\beta = |\mathrm{VR}(\{0,2\},\{1,3\})|/N,$$
$$\gamma = |\mathrm{VR}(\{0,3\},\{1,2\})|/N,$$

where $N = |\mathrm{VR}(\{0,1\},\{2,3\})| + |\mathrm{VR}(\{0,2\},\{1,3\})| + |\mathrm{VR}(\{0,3\},\{1,2\})|$. Note that $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma = 1$.

We will investigate what values of (α,β,γ) can be achieved by 4-regular gadgets. The motivation mainly comes from the question of whether a vertex in 4-regular maps can be simulated (exactly or approximately) by a 4-regular graph gadget. We omit the proofs in this section since they are lengthy, readers can refer to [10] for details.

We will discuss the power of 4-regular maps and 4-regular graphs separately. Before that, we introduce an operation 2-glue on gadgets. Given two gadgets G_1 and G_2, the 2-*glue* of G_1 and G_2 is a new gadget G_3 where half-edge 3 and 2 of G_1 are connected with half-edge 0 and 1 of G_2, respectively; and half-edge 0 and 1 of G_1 and half-edge 2 and 3 of G_2 are half-edges of G_3. The 2-glue operation is illustrated in Fig. 7.

The following theorem shows that 4-regular map gadgets can achieve almost all rational points (α,β,γ) on the plane $\alpha+\beta+\gamma = 1$ and $\alpha,\beta,\gamma \geq 0$.

Theorem 5 [10] *For every* $\alpha,\beta,\gamma \in \mathbb{Q}$ *such that* $0 \leq \alpha,\beta,\gamma < 1$ *and* $\alpha+\beta+\gamma = 1$, *there is a* 4*-regular map gadget having signature* (α,β,γ).

For 4-regular graph gadgets, let S contains of vectors $P(\alpha,\beta,\gamma)^{\mathrm{T}}$, where P is a 3×3 permutation matrix, $\alpha \geq \beta \geq \gamma \geq 0$, $\alpha+\beta+\gamma = 1$ and $\gamma \geq f(\beta)$, where $f(x) = (1-x)(1-\exp(2x/(x-1)))/2$. We illustrate S in Fig. 8. We first show that signatures in S are achievable by 4-regular graph gadgets:

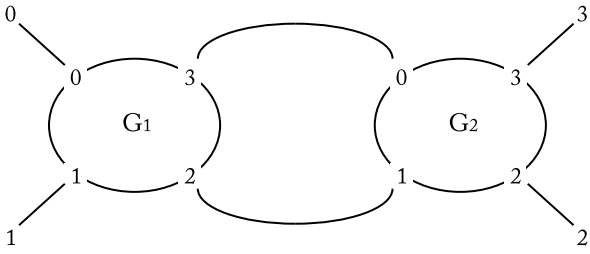


Fig. 7 2-glue of G_1 and G_2. The half-edges labeled *2* and *3* of G_1 are connected with half-edges labeled *1* and *0* of G_2, respectively. The *outmost straight lines* are the half-edges of the 2-glue of G_1 and G_2

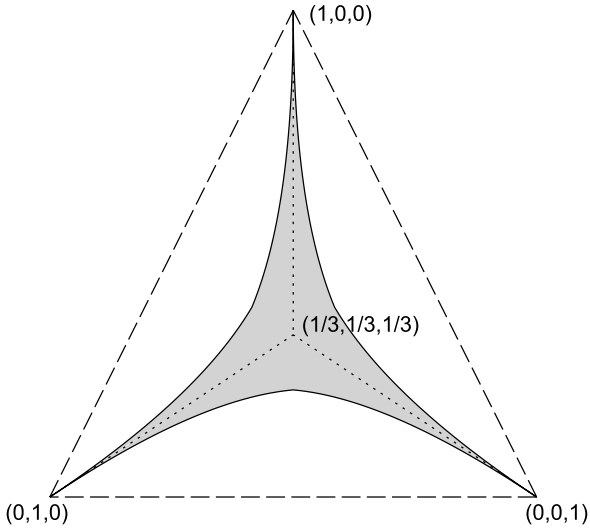


Fig. 8 The *shaded region S* contains the signatures which can be achieved by 4-regular graph gadgets as described in Theorem 6. The *triangle region within dashed lines* contains signatures (α, β, γ) s.t. $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$

Theorem 6 [10] *For every* $\mathbf{s} \in S$, *and for every* $\varepsilon > 0$, *there is a* 4*-regular graph gadget with signature* $\mathbf{s}'$ *such that*

$$\|\mathbf{s} - \mathbf{s}'\|_1 \leq \varepsilon.$$

On the other hand, we show S is closed under 2-glue operations.

Theorem 7 [10] *For* $i = 1, 2$, *let* G_i *be a* 4*-regular graph gadget with signature* $(\alpha_i, \beta_i, \gamma_i) \in S$. *Let* G_3 *be the* 2*-glue of* G_1 *and* G_2 *with signature* $(\alpha_3, \beta_3, \gamma_3)$, *hence* $(\alpha_3, \beta_3, \gamma_3) \in S$.

We performed experiment on all gadgets up to 7 vertices with random signature from S for each vertex, the result was in S. It seems that S is the largest region we can get for the signatures of 4-regular graph gadgets. Based on the results of our experiment, we conjecture that S contains all signatures of 4-regular graph gadgets.

Conjecture 1 *For every* 4*-regular graph gadget with signature* $\mathbf{s}$, $\mathbf{s} \in S$.

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